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Vorticity Generation on a Flat Surface in 3D Flows

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Vortex methods, based on the splitting into Euler and Stokes operators, have been successfully adopted in numerical solutions of three-dimensional Navier–Stokes equations in free-space. Here we deal with their application to flows bounded by solid walls, discussing in particular the boundary conditions for vorticity and their approximation. In two dimensions this has been accomplished by introducing a vortex sheet at the wall, determined by the local slip-velocity, as an approximation of the vorticity source. For three-dimensional flows, we analyze in the context of the Stokes substep the integral equation for the vorticity source and its connection with the creation algorithm adopted in vortex methods. The present analysis leads to a formulation which shows the connection between the exact vorticity source at the wall and the discrete vorticity creation operator adopted in the Chorin–Marsden formula. In particular, the slip velocity at the wall is identified as an approximate solution of the integral equation for the vorticity source and the corresponding error estimate is also discussed. Besides showing the consistency of this approximation, we indicate a numerical procedure which provides a wall-generation of solenoidal vorticity. This is a crucial issue for an accurate application of vortex methods to three-dimensional flows.

1. INTRODUCTION

In two-dimensional flows the mechanism of vorticity generation at a solid boundary is conveniently described in terms of a vortex sheet generated at the wall, which is diffused by viscosity in the interior of the flow domain. More specifically, for a given vorticity distribution, the Biot–Savart law provides a velocity field which, in general, violates the boundary conditions on both the normal and the tangential component. A potential flow is then required to enforce zero normal velocity. After this velocity component is added, the resulting flow still presents a slip velocity at the wall, which provides a vortex sheet having the exact intensity to bring the fluid particles at rest with respect to the solid boundary. In its turn, the vortex sheet, by diffusing, introduces new vorticity in the flow field. This mechanism, originally described by Lighthill [15] in physical terms, has been extensively used in numerical solutions of the incompressible Navier–Stokes equations for two-dimensional flows (see, e.g., Koumoutsakos and Leonard [14]), since Chorin introduced, in the context of vortex methods, the operator splitting technique [8]. In fact, the Chorin–Marsden product formula [9], besides factoring the convective and the diffusive components of the Navier–Stokes equations, considers a creation operator to introduce the concentrated vorticity layer at the wall which is successively diffused into the flow field. Afterwards, Benfatto and Pulvirenti provided a rigorous mathematical basis for this procedure. In particular in a first paper [6], they introduced the concept of a vorticity source at the wall, whose intensity is to be determined by solving a suitable integral equation. The integral equation reflects the nonlocal nature of the boundary condition for a vorticity formulation, as clearly pointed out by Quartapelle [16] in the context of a different numerical approach. In a successive paper [7] they show how, after introducing the local operator for the vorticity creation at the wall, the algorithm is convergent to the Navier–Stokes solution, obtained with the exact vorticity source.

In the present paper we study the generation by solid walls in three-dimensional flows with particular attention to the enforcement of wall boundary conditions in the context of viscous vortex methods. This numerical procedure, as in the two-dimensional case, requires the Navier–Stokes operator to be factored into two successive steps, one purely convective and the other purely diffusive, according to the Euler and Stokes equation, respectively. Although encouraging numerical results have been recently produced, several technical difficulties emerge when trying to prove the convergence of the splitting for three-dimensional flows, due in part to the lack of an existence theorem for the solution in the large of the Euler equations. Since this point has been analyzed, at least for flows in bounded domains, by Beale and Greengard in a recent paper [4], we concentrate on the Stokes step and, in particular, on the approximation of the boundary conditions. Following Cottet [11], we deal with a semi-infinite domain bounded by a planar surface. The simple geometry introduces several nice features, such as the possibility to enforce exactly part of the boundary conditions by considering a suitable extension of the solution in the whole space.

Let us denote by π the solid plane at $x_3 = 0$ and by $\mathbb{R}_3^+$ the three-dimensional upper halfspace. The evolution equation for the vorticity in a Stokes flow is then simply

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$$\frac{\partial \boldsymbol{\omega}}{\partial t} = \nu \Delta \boldsymbol{\omega}, \quad \boldsymbol{x} \in \mathbb{R}_3^+. \tag{1}$$

This equation requires suitable initial conditions, $\boldsymbol{\omega}_0$, together with Dirichlet boundary data

$$\boldsymbol{\omega}|_\pi = \boldsymbol{h} \tag{2}$$

or Neumann boundary data

$$\left.\frac{\partial \boldsymbol{\omega}}{\partial n}\right|_\pi = \boldsymbol{f}. \tag{3}$$

We assume, in particular, Dirichlet data for the normal component of vorticity and Neumann data for the two tangent components. With this choice h_n is directly inferred from the velocity at the boundary, since $h_n = \boldsymbol{n} \cdot \nabla \times \boldsymbol{u}$ only involves derivatives in the two directions tangent to π, and the projection $\boldsymbol{f}_\pi$ of the Neumann data on the plane corresponds to two unknown scalar functions. In any case the complete set of boundary conditions do not follow directly from the physical constraints on the corresponding velocity field $\boldsymbol{u}$, as discussed, among others, by Anderson [1]. Actually the Stokes operator requires the vanishing of both the normal and the tangential velocity component at the solid boundary. Either of these conditions is easily imposed directly on the velocity field which is related to the vorticity through the Poincaré representation formula [3],

$$\begin{aligned}\boldsymbol{u}^* = -\nabla_* \int_\pi (\boldsymbol{u} \cdot \boldsymbol{n})\, g dS + \nabla_* \times \int_\pi (\boldsymbol{n} \times \boldsymbol{u})\, g dS \\ + \nabla_* \times \int_{\mathbb{R}_3^+} \boldsymbol{\omega}\, g dV.\end{aligned} \tag{4}$$

where the normal $\boldsymbol{n}$ is directed towards the interior of the flow domain and $g = 1/(4\pi|\boldsymbol{x} - \boldsymbol{x}^*|)$ is the fundamental solution of the operator $-\Delta$.

In the limit as $\boldsymbol{x}^* \to \pi$, we obtain two different, although equivalent, boundary integral equations, by projecting on the normal and on the tangent plane, which allow us to determine either component of the velocity in terms of the other. The natural approach would be to consider the one obtained by the projection on the tangent plane and to enforce $\boldsymbol{u} \cdot \boldsymbol{n} = 0$ at the wall, leaving the unknown $\boldsymbol{n} \times \boldsymbol{u}$ to be expressed, via the integral equation, as a functional of the vorticity field $\boldsymbol{\omega}$. Typically, at this point, the no-slip condition is violated for a generic vorticity field, unless the further condition $\boldsymbol{n} \times \boldsymbol{u} = 0$ is used to obtain a physically acceptable solution. More specifically, a unique $\boldsymbol{\omega}$ is defined, once the boundary data are given, and, in the present case, we would have to determine $\boldsymbol{f}_\pi$ such that no-slip is satisfied during the whole evolution. To this purpose, $\boldsymbol{\omega}$ may be found as a functional of the data $\boldsymbol{f}_\pi$ through the integral representation for the solution of Eq. (1),

$$\boldsymbol{\omega}^* = \nu \int_0^t \int_\pi \left(\boldsymbol{\omega} \frac{\partial F}{\partial n} - F \frac{\partial \boldsymbol{\omega}}{\partial n} \right) dS\, d\tau + \int_{\mathbb{R}_3^+} \boldsymbol{\omega}_0 F_0\, dV, \tag{5}$$

where $F = 1/(4\pi\ \nu(t - \tau))^{(3/2)} e^{-|\boldsymbol{x}-\boldsymbol{x}^*|^2}/(4\nu(t - \tau))$ denotes the fundamental solution for the heat equation and $F_0 := F|_{\tau=0}$. In the limit as the field point approaches the boundary, a vector integral equation is obtained which determines the Dirichlet boundary values $\boldsymbol{h}_\pi$ from the corresponding values $\boldsymbol{f}_\pi$ and f_n in terms of the known value h_n. Consequently both the vorticity field $\boldsymbol{\omega}$ and the resulting velocity $\boldsymbol{u}$ are themselves functionally dependent only on $\boldsymbol{f}_\pi$. After combining this general solution for $\boldsymbol{\omega}$ with the representation (4) we find a vector boundary integral equation for the unknown data $\boldsymbol{f}_\pi$ by enforcing no-slip at the wall.

Even though the sketched procedure leads to a direct physical interpretation of the wall vorticity creation as introduced by Lighthill and Chorin, it is not convenient from a theoretical point of view. In particular, normal derivatives of the kernel functions appear in the equations, leading to operators which are external to the plane π. This drawback, already discussed in [6] for two-dimensional flows, complicates significantly the analysis of the resulting boundary integral equation for the vorticity source.

For this reason we follow a different procedure which, after a suitable extension of the field to the whole space, involves tangential operators and does not require normal derivatives. In particular, by this extension we obtain a representation for velocity which enforces first $\boldsymbol{n} \times \boldsymbol{u} = 0$ while the vorticity field is directly expressed in terms of the unknown $\boldsymbol{f}_\pi$ and of the initial vorticity $\boldsymbol{\omega}_0$. The vorticity source is finally determined as the solution of the integral equation following from the condition $\boldsymbol{n} \cdot \boldsymbol{u} = 0$. In three dimensions $\boldsymbol{f}_\pi$ is a two-dimensional vector and we discuss in the following how a scalar equation may determine this vector unknown. In particular, we obtain in Section 2 the relevant forms of the integral representations for velocity and vorticity, respectively. We enforce then the scalar constraint $\boldsymbol{u} \cdot \boldsymbol{n} = 0$ and we obtain in Section 3 the integral equation for the vorticity production. The exact solution of the equation is given in Section 4 and its approximation in terms of the slip velocity is discussed in the successive section 5, where an error estimate for the vorticity production during a time step is provided. Finally we briefly discuss in the last section the relevance of the present results to the more general Euler–Stokes splitting in three dimensions. Whenever possible, technical details have been avoided in the text by adding several appendices where all the calculations and the major assumptions for the error estimates are described.

2. INTEGRAL REPRESENTATIONS IN THE HALF-SPACE

When considering the upper half-space $\mathbb{R}_3^+$, we may directly obtain the boundary integral equation for the unknown $\boldsymbol{f}_\pi$. To this aim, suitable representations for velocity and vorticity are obtained by introducing the operator S, which by acting on a generic vector produces its image with respect to the plane π,

$$S_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}. \tag{6}$$

For convenience a superscript + will denote in the following a vector field in the physical domain $\mathbb{R}_3^+$, while a − will be used for the corresponding one in the complementary domain $\mathbb{R}_3^-$. In $\mathbb{R}_3^-$ we introduce the extension of the physical field $\boldsymbol{u}^+$,

$$\boldsymbol{u}^-(\boldsymbol{x}) = -S\boldsymbol{u}^+(S\boldsymbol{x}), \quad \boldsymbol{x} \in \mathbb{R}_3^-,$$

with corresponding vorticity

$$\boldsymbol{\omega}^- := \nabla \times \boldsymbol{u}^- = S\boldsymbol{\omega}^+(S\boldsymbol{x}), \quad \boldsymbol{x} \in \mathbb{R}_3^-. \tag{7}$$

The combined field, $\boldsymbol{u} := \boldsymbol{u}^+ \cup \boldsymbol{u}^-$, with vorticity $\boldsymbol{\omega} := \boldsymbol{\omega}^+ \cup \boldsymbol{\omega}^-$, may be expressed in terms of the Poincare' representation by

$$\boldsymbol{u}^* = -\nabla_* \int_\pi [\boldsymbol{u}] \cdot \boldsymbol{n} g dS + \nabla_* \times \int_\pi \boldsymbol{n} \times [\boldsymbol{u}]\, g\, dS + \nabla_* \times \int_{\mathbb{R}_3} \boldsymbol{\omega}\, g\, dV.$$

The jumps at π are accordingly defined as $[\boldsymbol{u}] = \boldsymbol{u}^+ - \boldsymbol{u}^-$, and, by the extension S used to define $\boldsymbol{u}$, it is easily shown that

$$\boldsymbol{n} \cdot [\boldsymbol{u}] = 0,$$
$$\boldsymbol{n} \times [\boldsymbol{u}] = 2(\boldsymbol{n}^+ \times \boldsymbol{u}),$$
$$\boldsymbol{\omega}(\boldsymbol{x}) = \boldsymbol{\omega}^+(\boldsymbol{x}) \cup S\boldsymbol{\omega}^+(S\boldsymbol{x}).$$

Hence we are left with the only condition $\boldsymbol{n} \times \boldsymbol{u} = 0$ to be enforced at π, and the above representation reduces to

$$\boldsymbol{u}^* = \nabla_* \times \int_{\mathbb{R}_3} \boldsymbol{\omega}\, g dV. \tag{8}$$

When $\boldsymbol{x}^*$ is on the boundary π, we find

$$u^*_{n|_\pi} = u^*_{3|_\pi} = 2 \int_{\mathbb{R}_3^+} \frac{1}{4\pi |\boldsymbol{x} - \boldsymbol{x}^*|^3} [(\boldsymbol{x} - \boldsymbol{x}^*) \times \boldsymbol{\omega}^+]_3\, dV,$$

$$u^*_{1|_\pi} = 0, \quad u^*_{2|_\pi} = 0,$$

where the symmetry properties of the field have been exploited to restrict the integration domain to the upper halfspace. Hence representation (8) does satisfy the no-slip, but in general implies a nonzero flux through the boundary π.

As concerning the vorticity, from the integral representation (5) for the fields $\boldsymbol{\omega}^+$ and $\boldsymbol{\omega}^-$ we arrive at

$$\boldsymbol{\omega}^* = \nu \int_0^t \int_\pi \left([\boldsymbol{\omega}] \frac{\partial F}{\partial n} - F \left[\frac{\partial \boldsymbol{\omega}}{\partial n} \right] \right) dS\, d\tau + \int_{\mathbb{R}_3} \boldsymbol{\omega}_0\, F_0\, dV, \tag{9}$$

where the initial conditions are $\boldsymbol{\omega}_0 = \boldsymbol{\omega}_0^+ \cup \boldsymbol{\omega}_0^-$ and the jumps

$$[\boldsymbol{\omega}] = \boldsymbol{\omega}^+_{|_\pi} - \boldsymbol{\omega}^-_{|_\pi}, \quad \left[\frac{\partial \boldsymbol{\omega}}{\partial n} \right] = \left. \frac{\partial \boldsymbol{\omega}^+}{\partial n} \right|_\pi - \left. \frac{\partial \boldsymbol{\omega}^-}{\partial n} \right|_\pi$$

reduce in the present case to

$$[\boldsymbol{\omega}] = \begin{cases} 0 \\ 0 \\ 2\omega^+_{3|_\pi} \end{cases} \quad \left[\frac{\partial \boldsymbol{\omega}}{\partial n} \right] = \begin{cases} 2\dfrac{\partial \omega_1}{\partial n} \\ 2\dfrac{\partial \omega_2}{\partial n} \\ 0. \end{cases}$$

By recalling the definition for $\boldsymbol{h}$ and $\boldsymbol{f}$ given in Section 2, the representation (9), combined with the jump relations and the condition $h_3 \equiv h_n = 0$ (following from zero slip velocity), yields

$$\boldsymbol{\omega}^* = -2\nu \int_0^t \int_\pi F \boldsymbol{f}_\pi\, dS\, d\tau + \int_{\mathbb{R}_3} \boldsymbol{\omega}_0\, F_0\, dV, \tag{10}$$

where the initial conditions,

$$\boldsymbol{\omega}_0(\boldsymbol{x}) = \boldsymbol{\omega}_0^+(\boldsymbol{x}) \cup S\boldsymbol{\omega}_0^+(S\boldsymbol{x}), \tag{11}$$

give the zero normal component at the wall,

$$\int_{\mathbb{R}_3} \omega_{03}\, F_0\, dV|_{\mathbf{x}^* \in \pi} = 0, \tag{12}$$

as it follows from symmetry considerations.

It is worth stressing that, in the above form, the normal component f_3 does not appear explicitly. In fact, we used the Dirichlet data $h_3 = 0$ and f_3 may be evaluated *a posteriori* from the representation, once the actual unknown, the two-dimensional vector $\boldsymbol{f}_\pi$, has been determined. Concluding the section, we recall that only a single scalar condition, namely $\boldsymbol{u} \cdot \boldsymbol{n} = 0$, has to be still imposed to the field. How to use this constraint to determine the vector unknown $\boldsymbol{f}_\pi$ is illustrated in the following section. We only note here, the details are in Appendix D, that

$$\nabla_* \cdot \boldsymbol{\omega}^*(\boldsymbol{x}^*, t) = -2\nu \int_0^t d\tau \int_\pi \nabla_\pi \cdot \boldsymbol{f}_\pi F \, dS + \int_{\mathbb{R}_3} \nabla \cdot \boldsymbol{\omega}_0 F_0 \, dV;$$

hence, for $\nabla \cdot \boldsymbol{\omega}_0 = 0$, representation (10) yields a solenoidal field, provided that $\nabla_\pi \cdot \boldsymbol{f}_\pi = 0$.

3. BOUNDARY INTEGRAL EQUATION FOR THE VORTICITY SOURCE

Equation (10) states that the field $\boldsymbol{\omega}$ at time t follows from the diffusion of the initial field $\boldsymbol{\omega}_0$ and from the continuous local introduction of new vorticity provided by the wall source $\boldsymbol{f}_\pi$. The intensity of the vector source of vorticity has to be determined in such a way as to satisfy zero normal velocity at the wall. The required equation is obtained by simply introducing Eq. (10) into the representation (8) for the velocity field. By taking the limit as the field point approaches π and by projecting along the normal $\boldsymbol{n}$, after enforcing $\boldsymbol{u} \cdot \boldsymbol{n} = 0$, we obtain in fact a scalar boundary integral equation,

$$\begin{aligned} &\boldsymbol{n}^* \cdot \nabla_* \times \int_{\mathbb{R}_3} \left(2\nu \int_0^t \int_\pi F \boldsymbol{f}_\pi \, dS \, d\tau \right) g \, dV \\ &\quad = \boldsymbol{n}^* \cdot \nabla_* \times \int_{\mathbb{R}_3} \left(\int_{\mathbb{R}_3} \boldsymbol{\omega}_0 F_0 \, dV \right) g dV, \end{aligned} \tag{13}$$

for the vector unknown $\boldsymbol{f}_\pi$. Only the projection of $\boldsymbol{\omega}_0$ onto the plane π and tangential derivatives are actually involved in the right-hand side of (13). By denoting by $\boldsymbol{\omega}_{\pi_0}(\boldsymbol{x})$ the projection of the initial vorticity on the plane wall, after introducing the position

$$\tilde{\boldsymbol{\omega}}_\pi := \int_{\mathbb{R}_3} \boldsymbol{\omega}_{\pi_0} F_0 \, dV,$$

we note, as a consequence of the introduced symmetry, (6), that

$$\tilde{\boldsymbol{\omega}}_\pi^-(\boldsymbol{x}) = \tilde{\boldsymbol{\omega}}_\pi^+(\boldsymbol{x})(S\boldsymbol{x}).$$

Hence the integral on the right-hand side of Eq. (13), evaluated at boundary points, is exactly given by twice the contribution arising from the upper halfspace. This property is maintained under the action of the tangential operator $\boldsymbol{n}^* \cdot \nabla_* \times (\cdot)$, and, by considering the restriction to the upper halfspace of the vorticity arising from the diffusion of the initial field, we may simplify the writing of the integral equation to enhance its interpretation,

$$\begin{aligned} &\boldsymbol{n}^* \cdot \nabla_* \times \int_{\mathbb{R}_3} \left(\nu \int_0^t \int_\pi F \boldsymbol{f}_\pi \, dS \, d\tau \right) g \, dV \\ &\quad = \boldsymbol{n}^* \cdot \nabla_* \times \int_{\mathbb{R}_3^+} \left(\int_{\mathbb{R}_3} \boldsymbol{\omega}_0 F_0 \, dV \right) g dV. \end{aligned} \tag{14}$$

We may note that Eq. (14) is substantially a three-dimensional extension of the corresponding equation considered by Benfatto and Pulvirenti in [6]. In fact they introduced the use of the vorticity source to enforce the boundary condition; however, the intrinsic connection between the wall-source and the slip velocity was not entirely exploited. To this purpose we may express the right-hand side of (14) as

$$\begin{aligned} &\boldsymbol{n}^* \cdot \nabla^* \times \int_{\mathbb{R}_3^+} \left(\int_{\mathbb{R}_3} \boldsymbol{\omega}_0 F_0 \, dV \right) g \, dV \\ &\quad = -\boldsymbol{n}^* \cdot \nabla^* \times \int_\pi (\boldsymbol{n} \times \boldsymbol{u}_S) \, g dS, \end{aligned} \tag{15}$$

in terms of the slip velocity $\boldsymbol{n} \times \boldsymbol{u}_S$ associated with the field $\boldsymbol{u}_S$ that would originate from the diffusion of the initial vorticity $\boldsymbol{\omega}_0$, according to the field extension (11). This *virtual* field is requested to satisfy the boundary condition $\boldsymbol{u}_S \cdot \boldsymbol{n} = 0$ and, in Chorin's method, it provides the concentrated vortex sheet at the wall whose diffusion introduces the new vorticity required for an approximate satisfaction of the no-slip condition. By using this quantity we reexpress Eq. (14) as

$$\begin{aligned} &\boldsymbol{n}^* \cdot \nabla_* \times \int_{\mathbb{R}_3} \left(\nu \int_0^t \int_\pi F \boldsymbol{f}_\pi \, dS \, d\tau \right) g \, dV \\ &\quad = -\boldsymbol{n}^* \cdot \nabla^* \times \int_\pi (\boldsymbol{n} \times \boldsymbol{u}_S) \, g dS. \end{aligned} \tag{16}$$

Equation (16) captures the essence of Chorin's vorticity creation concept. Since

$$\int_0^t \int_\tau F \boldsymbol{f}_\pi \, dS \, d\tau \rightarrow \int_0^t \boldsymbol{f}_\pi \, d\tau \, \delta(x_3), \quad t \rightarrow 0,$$

it follows immediately that

$$\nu \int_0^t \boldsymbol{f}_\pi \, d\tau \approx -\boldsymbol{n} \times \boldsymbol{u}_S \quad \text{for very short times.} \tag{17}$$

Physically Eq. (17) indicates that the amount of vorticity introduced in one time step is essentially given by the vortex sheet with intensity $-\boldsymbol{n} \times \boldsymbol{u}_S$, a well-known and largely discussed result for two-dimensional flows (see, e.g., Koumotsakos, Leonard, and Pepin [13]). In the following, we discuss the nature of this *ansatz* which completely parallels, in three dimensions, Chorin's approximation for two-dimensional flows. To discuss this point in further detail, let us address first how the scalar equation (16) can handle the vector unknown $\boldsymbol{f}_\pi$. To this purpose we adopt a Hodge-type decomposition [3] for the two tangent vector fields $\boldsymbol{u}_S$ and $\boldsymbol{f}_\pi$,

$$\boldsymbol{u}_S = \nabla_\pi \phi_u + (\boldsymbol{n} \times \nabla)_\pi \psi_u \tag{18}$$

$$\boldsymbol{f}_\pi = (\nabla_\pi \phi_f + (\boldsymbol{n} \times \nabla)_\pi \psi_f) \times \boldsymbol{n}, \tag{19}$$

where the vector product with the normal $\boldsymbol{n}$ in the second equation has been merely introduced for convenience. It may be noted that the above positions completely correspond, for the present flat boundary, to the Helmholtz decomposition of a vector field in $\mathbb{R}^2$ in terms of a scalar and a vector potential. After applying the operator $\boldsymbol{n} \cdot \nabla \times (\cdot)$ to Eq. (18), we obtain the Laplace–Beltrami equation $\Delta_\pi \psi_u = \boldsymbol{n} \cdot \nabla \times (\boldsymbol{u}_S)$, which in the present case reduces to a Poisson equation on the plane π. By the definition of the *virtual* field $\boldsymbol{u}_S$ and the result given in Eq. (12),

$$\boldsymbol{n} \cdot \nabla \times \boldsymbol{u}_S = \boldsymbol{n} \cdot \left(\int_{\mathbb{R}_3} \boldsymbol{\omega}_0 \, F_0 \, dV \right)\Big|_\pi \equiv 0;$$

hence, from the Poisson equation we have $\psi_u = 0$. Concerning the wall source, as discussed in the previous section, to obtain a solenoidal vorticity we must enforce $\nabla_\pi \cdot \boldsymbol{f}_\pi = 0$. Consistently, from decomposition (19) we have $\Delta_\pi \psi_f = 0$; hence $\psi_f = 0$. We may state this intermediate result as follows: both the wall source $\boldsymbol{f}_\pi$ and the wall slip velocity $\boldsymbol{u}_S$ can be expressed in terms of irrotational vector fields on the plane π with potential ϕ_f and ϕ_u, respectively. The actual unknown, ϕ_f, has to be found from Eq. (16) rewritten as

$$\begin{aligned} \boldsymbol{n}^* \cdot \nabla_* \times \int_{\mathbb{R}_3} \nu \left(\int_0^t \int_\pi F(\boldsymbol{n} \times \nabla_\pi) \phi_f \, dS \, d\tau \right) g \, dV \\ = \boldsymbol{n}^* \cdot \nabla_* \times \int_\pi (\boldsymbol{n} \times \nabla)_\pi \phi_u \, g \, dS, \end{aligned} \tag{20}$$

which is a scalar equation for a single scalar unknown. The exact solution is achieved in the next section, as the basis for a more precise discussion of Chorin's approximation and of the related error estimate, finally provided in Section 5.

4. SOLUTION OF THE INTEGRAL EQUATION

We give here the exact solution of Eq. (20) for the wall source. After introducing the notation

$$\Phi_f := \int_0^t \phi_f \, d\tau, \tag{21}$$

we obtain the two-dimensional Fourier transform,

$$\check{\Phi}_f(\boldsymbol{\xi}, t) = \frac{1}{2\pi} \int_{\mathbb{R}^2} d\boldsymbol{\xi} \, \Phi_f(\boldsymbol{X}, t) e^{-\imath \boldsymbol{\xi} \cdot \boldsymbol{X}},$$

of the physical solution Φ_f, which may follow by inverse transform. However, Φ_f is not reported here explicitly since, as shown in the following section, the error related to approximation (17) is better analyzed by considering directly its Fourier transform.

We introduce the integration by parts,

$$\begin{aligned} &\int_0^t F(\boldsymbol{x}, t-\tau) \phi_f(\boldsymbol{Y}, \tau) \, d\tau \\ &\quad = \Phi_f(\boldsymbol{Y}, t) \, \delta(\boldsymbol{x}) + \int_0^t \Phi_f(\boldsymbol{Y}, \tau) F_t(\boldsymbol{x}, t-\tau) \, d\tau, \end{aligned}$$

to be understood in the sense of distributions, where we used $F_\tau = -F_t$ and the subscripts t and τ denote differentiation with respect to the corresponding time variable. After introducing this result into Eq. (20) we obtain

$$\begin{aligned} \boldsymbol{n}^* \cdot \nabla_* \times \int_\pi \nu g (\boldsymbol{n} \times \nabla_\pi) \Phi_f \, dS + \boldsymbol{n}^* \cdot \nabla_* \times \boldsymbol{e}^* \\ = \boldsymbol{n}^* \cdot \nabla_* \times \int_\pi (\boldsymbol{n} \times \nabla)_\pi \, \phi_u \, g \, dS, \end{aligned} \tag{22}$$

where

$$\begin{aligned} \boldsymbol{e}(\boldsymbol{x}^*, t) := \nu \int_0^t d\tau \int_\pi d\boldsymbol{Y} \boldsymbol{n} \times \nabla_\pi^Y \Phi_f(\boldsymbol{Y}, \tau) \\ \cdot \int_{\mathbb{R}^3} g(\boldsymbol{x}^* - \boldsymbol{x}) F_t(\boldsymbol{x} - \boldsymbol{Y}, t - \tau) \, d\boldsymbol{x}. \end{aligned} \tag{23}$$

The integral over $\mathbb{R}^3$ is the three-dimensional convolution, $g * F_t$, which is readily Fourier transformed (in three dimensions) as $(2\pi)^{3/2} \hat{g}(\kappa) \cdot \hat{F}_t(\kappa, t-\tau)$. Here the transform of the fundamental solution of the Laplace equation is $\hat{g} = 1/((2\pi)^{(3/2)} \kappa^2)$ and for the time derivative of the free-space Green's function for the heat equation we have $\hat{F}_t = -\nu \kappa^2 \hat{F}_3(\kappa, t-\tau)$ (the suffix 3 has been added to F to recall that we are dealing with the three-dimensional fundamental solution). Hence

$$\hat{g}(\kappa) \cdot \hat{F}_t(\kappa, t-\tau) = -\frac{\nu}{(2\pi)^{(3/2)}} \hat{F}_3(\kappa, t-\tau),$$

and by taking the inverse Fourier transform we find

$$g * F_t = -\nu F_3(\boldsymbol{x}^* - \boldsymbol{Y}, t - \tau).$$

When this expression is introduced into Eq. (23),

$$\boldsymbol{e}(\boldsymbol{x}^*, t) = -\nu^2 \int_\pi d\boldsymbol{Y} \int_0^t d\tau (\boldsymbol{n} \times \nabla_\pi^Y)\Phi_f(\boldsymbol{Y}, \tau) \cdot F_3(\boldsymbol{x}^* - \boldsymbol{Y}, t - \tau),$$

after recalling that for $\boldsymbol{x}^* \in \pi$ we may express F_3 in terms of the two-dimensional fundamental solution of the heat-equation, $F_2 = 1/(4\pi\nu(t-\tau))e^{-|\boldsymbol{x}^*-\boldsymbol{Y}|/4\nu(t-\tau)}$, as $F_3 = 1/\sqrt{4\pi\nu(t-\tau)}\, F_2(\boldsymbol{x}^* - \boldsymbol{Y}, t - \tau)$, we obtain for $\boldsymbol{e}$,

$$\boldsymbol{e}(\boldsymbol{x}^*, t) = -\nu^2 \int_\pi d\boldsymbol{Y} \int_0^t d\tau \frac{1}{\sqrt{4\pi\nu(t-\tau)}} (\boldsymbol{n} \times \nabla_\pi^Y)\Phi_f(\boldsymbol{Y}, \tau) \cdot F_2(\boldsymbol{x}^* - \boldsymbol{Y}, t - \tau).$$

This is the form required to perform the successive steps in the analysis of the integral equation (22). First, all the terms appearing in the equation, when we ignore the time integrations and the dependence on t and τ, are of the kind

$$\boldsymbol{n}^* \cdot \nabla^* \times \int_\pi d\boldsymbol{Y}(\boldsymbol{n} \times \nabla_\pi^Y) q(\boldsymbol{Y}) K(\boldsymbol{x}^* - \boldsymbol{Y}).$$

Integrating by parts, after assuming the density q to suitably vanish for large $|\boldsymbol{Y}|$, we have

$$\boldsymbol{n}^* \cdot \nabla^* \times (\boldsymbol{n}^* \times \nabla^*) \int_\pi d\boldsymbol{Y}\, q(\boldsymbol{Y}) K(\boldsymbol{x}^* - \boldsymbol{Y}),$$

where we used $\boldsymbol{n} \times \nabla_\pi^Y K = -\boldsymbol{n}^* \times \nabla^* K$. The operator acting on the surface integral corresponds to the Laplace–Beltrami operator Δ_π, which for the present flat geometry corresponds to the Laplacian Δ_2:

$$\Delta_2^* \int_\pi d\boldsymbol{Y}\, q(\boldsymbol{Y}) K(\boldsymbol{x}^* - \boldsymbol{Y}).$$

Hence, we may write the integral equation (22) as

$$\Delta_2^* \left(g \star \Phi_f + c \int_0^t \frac{d\tau}{\sqrt{t-\tau}} (F_2 \star \Phi_f) \right) = \Delta_2^* \left(g \star \frac{\phi_u}{\nu} \right), \quad \boldsymbol{x}^* \in \pi, \tag{24}$$

where $c = -\frac{1}{2}\sqrt{\nu/\pi}$ and we denote by $a \star b$ the two-dimensional convolution $\int_\pi d\boldsymbol{Y}\, a(\boldsymbol{Y}) b(\boldsymbol{x}^* - \boldsymbol{Y})$, with $\boldsymbol{x}^* \in \pi$.

To proceed further we take the two-dimensional Fourier transform of Eq. (24),

$$|\boldsymbol{\xi}|^2 \check{g}(\boldsymbol{\xi}) \check{\Phi}_f(\boldsymbol{\xi}, t) + c|\boldsymbol{\xi}|^2 \int_0^t \frac{d\tau}{\sqrt{t-\tau}} \check{F}_2(\boldsymbol{\xi}, t-\tau) \check{\Phi}_f(\boldsymbol{\xi}, \tau) = |\boldsymbol{\xi}|^2 \check{g}(\boldsymbol{\xi}) \frac{\check{\phi}_u(\boldsymbol{\xi}, t)}{\nu}. \tag{25}$$

Since (Appendix A) we have

$$\check{g}(\boldsymbol{\xi}) = \frac{1}{4\pi} \frac{1}{|\boldsymbol{\xi}|}$$

and

$$\check{F}_2(\boldsymbol{\xi}, t-\tau) = \frac{1}{2\pi} e^{-\nu\xi^2(t-\tau)},$$

from (25) we obtain the integral equation of the second kind,

$$\check{\Phi}_f(\boldsymbol{\xi}, t) + \check{c}|\boldsymbol{\xi}| \int_0^t \frac{d\tau}{\sqrt{t-\tau}} e^{-\nu\xi^2(t-\tau)} \check{\Phi}_f(\boldsymbol{\xi}, \tau) = \frac{\check{\phi}_u(\boldsymbol{\xi}, t)}{\nu}, \tag{26}$$

where $\check{c} = -\sqrt{\nu/\pi}$ and $\xi = |\boldsymbol{\xi}|$. The corresponding solution, which is obtained by Laplace transforms (see Appendix B), may be written explicitly as

$$\check{\Phi}_f(\boldsymbol{\xi}, t) = \frac{\check{\phi}_u}{\nu}(\boldsymbol{\xi}, t) + \sqrt{\nu/\pi}\, |\boldsymbol{\xi}| \int_0^t d\tau \frac{e^{-\nu\xi^2(t-\tau)}}{\sqrt{t-\tau}} \frac{\check{\phi}_u}{\nu}(\boldsymbol{\xi}, \tau) + \nu |\boldsymbol{\xi}|^2 \int_0^t d\tau\, \mathrm{erfc}(-\sqrt{\nu}\, |\boldsymbol{\xi}|\, \sqrt{t-\tau}) \frac{\check{\phi}_u}{\nu}(\boldsymbol{\xi}, \tau), \tag{27}$$

and the solution in physical variables may be obtained by inverse Fourier transform.

Equation (27) gives $\check{\Phi}_f$ in terms of the Fourier transform of ϕ_u, the potential function for the slip velocity, $\boldsymbol{u}_S = \nabla_\pi \phi_u$. The structure of the exact solution already gives good insight about the approximation of the boundary conditions in terms of the slip velocity. We immediately observe that the error is related to a suitable norm of the difference $\nu\check{\Phi}_f - \check{\phi}_u$, which is proportional to time convolution integrals involving $\check{\phi}_u$. A more complete error analysis is performed in the next section.

5. THE WALL VORTEX SHEET AS AN APPROXIMATION OF THE VORTICITY SOURCE

We are now in a position to obtain one of the basic results of the paper, namely an estimate of the error introduced by the procedure adopted in numerical methods, where the approximate value given by the vortex sheet at the wall is used, instead of solving for the unknown vorticity source. We analyze first the case where the vortex sheet has an arbitrary intensity, although subject to certain requirements to be specified later. This is likely to be the relevant condition at the first time step of a discrete method, if, as usual, the vorticity field at $t = 0$ violates the no-slip condition at the wall. Afterwards we consider the vortex sheet as originating from the free-space diffusion of an initial vorticity field satisfying both $\boldsymbol{u} \cdot \boldsymbol{n} = 0$ and $\boldsymbol{n} \times \boldsymbol{u} = 0$, as it should be the usual case for successive time steps.

In particular, we are interested in estimating the error, expressed in physical variables, using the L^∞ norm,

$$E_\infty := \|\nu\boldsymbol{n}\times\nabla\Phi_f - \boldsymbol{n}\times\nabla\phi_u\|_{L^\infty} = \sup_{\mathbf{x}\in\pi}|\nu\boldsymbol{n}\times\nabla\Phi_f - \boldsymbol{n}\times\nabla\phi_u|,$$

which is immediately related to the L^1-norm of the corresponding Fourier transforms by simply recalling the inequality for a function q,

$$\|q\|_{L^\infty} \le c \int_{\mathbb{R}^2} |\check{q}(\boldsymbol{\xi})|\, d\boldsymbol{\xi} = c\|\check{q}\|_{L^1}.$$

Using this inequality we have

$$\begin{aligned}\|\nu\boldsymbol{n}\times\nabla\Phi_f - \boldsymbol{n}\times\nabla\phi_u\|_{L^\infty} &\le c \int_{\mathbb{R}^2} |\boldsymbol{\xi}|\,|\nu\check{\Phi}_f - \check{\phi}_u| d\boldsymbol{\xi} \\ &\le c_1 \int_0^t \frac{\nu d\tau}{\sqrt{\nu(t-\tau)}} \int_{\mathbb{R}^2} d\boldsymbol{\xi}|\boldsymbol{\xi}|^2|\check{\phi}_u(\boldsymbol{\xi},\tau)| \\ &\quad + c_2 \int_0^t \nu d\tau \int_{\mathbb{R}^2} d\boldsymbol{\xi}|\boldsymbol{\xi}|^3|\check{\phi}_u(\boldsymbol{\xi},\tau)|,\end{aligned}$$

where the last step follows from $e^{-\nu|\boldsymbol{\xi}|^2(t-\tau)} \le 1$, erfc $(-\sqrt{\nu}|\boldsymbol{\xi}|\sqrt{t-\tau}) \le 2$, and Eq. (27). The two dimensionless constants are $c_1 = 1/\sqrt{\pi}$ and $c_2 = 2$. If

$$\int_{\mathbb{R}^2} d\boldsymbol{\xi}|\boldsymbol{\xi}|^2|\check{\phi}_u(\boldsymbol{\xi},\tau)| \le D_2, \quad \int_{\mathbb{R}^2} d\boldsymbol{\xi}|\boldsymbol{\xi}|^3|\check{\phi}_u(\boldsymbol{\xi},\tau)|, \le D_3, \quad (28)$$

where D_2, $D_3 > 0$ denote two constants with different physical dimensions, it follows that for $t \to 0$

$$\|\nu\boldsymbol{n}\times\nabla\Phi_f - \boldsymbol{n}\times\nabla\phi_u\|_{L^\infty} = \mathcal{O}(\sqrt{\nu t}) \le 2c_1 D_2\sqrt{\nu t}. \quad (29)$$

Inequality (29), where the symbol $\mathcal{O}$ is assumed to have suitable dimensions for consistency with the left-hand side, gives the result we anticipated at the end of the previous section, i.e., the convergence, although at a slow rate, of the approximate solution, $\boldsymbol{n} \times \nabla\phi_u$, to the exact one, $\nu\boldsymbol{n} \times \nabla\Phi_f$, for arbitrary intensity of the wall vortex sheet. The error estimate is valid for a generic initial condition, in particular, the one involving a finite slip at the wall, as it may occur at the first time step; see Appendix C. The constant appearing in the error bound is readily estimated in terms of the Fourier transform $\check{\phi}_u$ of the velocity potential at the wall (at the wall $\boldsymbol{u}_S = \nabla_\pi\phi_u$, even for rotational flows; see Section 3). Consistently the present result may be of some practical use in the actual computations and also at the successive steps, when the slip velocity is vanishingly small. In these conditions the quantity D_2, which is correspondingly small, may be exploited to give a formal evaluation of the accuracy. Let us consider, for instance, the case when no-slip is exactly satisfied by the initial field at the beginning of the step. In this case we have (Appendix C)

$$D_2 = \mathcal{O}(\nu\tau), \quad D_3 = \mathcal{O}(\nu\tau), \quad (30)$$

where the proper dimensions are accounted for by the two symbols $\mathcal{O}$; hence we readily obtain for $t \to 0$

$$\begin{aligned}\|\nu\boldsymbol{n}\times\nabla\Phi_f - \boldsymbol{n}\times\nabla\phi_u\|_{L^\infty} &\le c_1 \int_0^t \frac{\mathcal{O}(\nu\tau)\,\nu\, d\tau}{\sqrt{\nu(t-\tau)}} \\ &\quad + c_2\int_0^t \mathcal{O}(\nu\tau)\,\nu\, d\tau = \mathcal{O}((\nu t)^{3/2});\end{aligned} \quad (31)$$

that is, the replacement of the exact vorticity source at the wall with the vortex sheet $\boldsymbol{n} \times \boldsymbol{u}_S$ leads to an error of order $\mathcal{O}(t^{3/2})$. We finally mention that the asymptotic behavior of the quantities D_2, D_3 defined in (28) is strictly related to smoothness properties of the initial field $\boldsymbol{\omega}_0$, as given by

$$M_d := \sup_z \int_{\mathbb{R}^2} |\boldsymbol{\xi}|^d|\check{\boldsymbol{\omega}}_0(\boldsymbol{\xi}, z)|\, d\boldsymbol{\xi} < \infty \quad \forall d = 0, ..., 3. \quad (32)$$

These assumptions require the vorticity field $\boldsymbol{\omega}_0$ to have certain derivatives in the directions parallel to the solid wall which can be expressed in terms of the corresponding Fourier transform. These derivatives in particular have to be bounded in the L^∞ norm for any distance z from the wall. As a consequence the quantities D_2 and D_3 defined by (28) are finite or vanishing, as illustrated in details in Appendix C. Hence the smoothness of $\boldsymbol{\omega}_0$ involves corresponding properties of the vortex sheet intensity. As a result, the error is an order $\sqrt{t}$ smaller than the strength of the vortex sheet induced at the wall.

Let us recall the main achievements of this section in terms of physical variables. When the slip condition is violated initially, we have an $\mathcal{O}(1)$ strength for the vortex sheet induced at the wall after diffusion, and the error, due to the approximate condition, is given by

$$\|\nu \int_0^t \boldsymbol{f}_\pi\, d\tau - u_S \times \boldsymbol{n}\|_{L^\infty} = M_0\, \mathcal{O}(\sqrt{\nu t}).$$

On the other hand, when the initial vorticity induces zero velocity at the wall, the slip after diffusion is small. Under these conditions the approximation introduces an error given by

$$\|\nu \int_0^t \boldsymbol{f}_\pi\, d\tau - u_S \times \boldsymbol{n}\|_{L^\infty} = M_2\, \mathcal{O}((\nu t)^{3/2}).$$

For a generic time step we could have to consider a residual effect of the finite vortex sheet which may occur, as discussed before, at the initial time. We expect that the error accumulation does not prevent the convergence of a multistep procedure for the Stokes flow, although a rigorous result on this point is beyond the scope of the present paper.

Let us finally note that the results obtained in the present section are, in a loose sense, essentially due to the nature of the kernels appearing in Eq. (16). Actually, the elliptic behavior induced by the kernel g, appearing in both sides, is eliminated, while the kernel F of the heat equation is highly localized for small times. As a consequence, the local mechanism in terms of slip velocity is a consistent approximation of the exact boundary conditions since it provides the amount of vorticity to enforce the no-slip condition at the solid wall in the limit of a vanishing time step.

6. CONCLUDING REMARKS

The relevance of the present results, concerning the vorticity generation in a Stokes flow, to the more general context of the Euler–Stokes splitting for the three-dimensional Navier–Stokes equations is here briefly analyzed. The convergence of the splitting procedures in three dimensions, although restricted to bounded domains, has been demonstrated in a recent paper [4] by Beale and Greengard through the use of the velocity-pressure formulation of the equations. Once the splitting has been established, the development of a corresponding numerical algorithm requires suitable definitions for the approximate Euler and Stokes flows, as is easily accomplished in the context of vortex methods. At least for flows in unbounded domains, this approach presents several advantages. Among others, computational resources are concentrated where they are required, e.g., the rotational region, and the far field behavior is exactly satisfied by the integral representation for the velocity field. Concerning the inviscid step, we rely on the technical achievements gained by the blob method [2, 5, 17] in the evolution analysis of free-space vorticity structures, without any major complication for the presence of the solid walls. When dealing with the viscous step in three dimensions, instead, we face a new difficulty for the definition of the proper approximation for the vorticity boundary condition at the solid wall.

The present extension of the integral equation which describes the solid wall as a source of vorticity establishes a direct connection between the wall source and the slip velocity at the wall. The exact solution of this equation allows us to identify the slip at the wall as a consistent approximation of the vorticity source. Hence, the boundary conditions can be enforced, via a local procedure, as in the original two-dimensional algorithm. When considering the splitting procedure for the single step, since the error is an order $\sqrt{t}$ smaller than the induced vortex sheet and the vortex sheet provided by the inviscid evolution must vanish with the time step, we argue that the error estimate should not be deteriorated. A complete proof of convergence for the full multistep Euler–Stokes procedure is not presently available for three dimensions.

As a further contribution of relevance for numerical applications, we have discussed in the context of the present analysis a proper way to obtain a solenoidal approximation of the wall source which would introduce divergence-free vorticity in the field after diffusion of the vortex sheet. This feature is highly recommended for a particle-based approximation of the vorticity in three dimensions, where the issue of maintaining an almost solenoidal discrete field may be crucial for a reliable flow simulation.

APPENDIX A

We give the expression for the two-dimensional Fourier transform of the fundamental solution of the three-dimensional Laplace operator. Considering a generic point $\boldsymbol{x} \in \mathbb{R}^3$, let us denote by $\boldsymbol{X}$ its projection onto π and by z its distance from the plane π. The two-dimensional Fourier transform of $g(\boldsymbol{Y} - \boldsymbol{x}) = 1/(4\pi\,|\boldsymbol{Y} - \boldsymbol{x}|)$, for $\boldsymbol{Y} \in \pi$,

$$\breve{g}(\boldsymbol{\xi}, \boldsymbol{x}) = \frac{1}{2\pi}\int_{\mathbb{R}^2} d\boldsymbol{Y} e^{-\iota\boldsymbol{\xi}\cdot\boldsymbol{Y}} g(\boldsymbol{Y} - \boldsymbol{x}),$$

may be written as

$$c e^{-\iota\boldsymbol{\xi}\cdot\boldsymbol{X}} \int_{\mathbb{R}^2} d\boldsymbol{Y}' \frac{e^{-\iota\boldsymbol{\xi}\cdot\boldsymbol{Y}'}}{\sqrt{|\boldsymbol{Y}'|^2 + z^2}} = c e^{-\iota\boldsymbol{\xi}\cdot\boldsymbol{X}} \int_{\mathbb{R}} dY_2 e^{-\iota\xi_2 Y_2} \int_{\mathbb{R}} dY_1 \frac{e^{-\iota\xi_1 Y_1}}{\sqrt{Y_1^2 + (Y_2^2 + z^2)}},$$

where $c = 1/(8\pi^2)$. From the known integrals [12]

$$\int_{\mathbb{R}} dY_1 \frac{e^{-\iota\xi_1 Y_1}}{\sqrt{Y_1^2 + (Y_2^2 + z^2)}} = 2K_0(\xi_1\sqrt{Y_2^2 + z^2})$$

and

$$\int_{\mathbb{R}} dY_2 e^{-\iota\xi_2 Y_2} K_0(\xi_1\sqrt{Y_2^2 + z^2}) = \pi\frac{e^{-|z|\,|\boldsymbol{\xi}|}}{|\boldsymbol{\xi}|},$$

where K_0 is the modified Bessel function of order zero, we may obtain

$$\breve{g}(\boldsymbol{\xi}, \boldsymbol{x}) = \frac{1}{4\pi} \frac{e^{-|z|\,|\boldsymbol{\xi}|}}{|\boldsymbol{\xi}|} e^{-\imath \boldsymbol{\xi}\cdot \boldsymbol{X}}.$$

For $\boldsymbol{X} = 0$ we have

$$\breve{g}(\boldsymbol{\xi}, \boldsymbol{x})_{|_{\boldsymbol{X}=0}} = \frac{1}{4\pi} \frac{e^{-|z|\,|\boldsymbol{\xi}|}}{|\boldsymbol{\xi}|}, \tag{33}$$

which, when $z = 0$, reduces to

$$\breve{g}(\boldsymbol{\xi}, \boldsymbol{x})_{|_{\boldsymbol{x}=0}} = \frac{1}{4\pi} \frac{1}{|\boldsymbol{\xi}|}. \tag{34}$$

APPENDIX B

We obtain the solution of Eq. (26), repeated here for convenience,

$$\begin{aligned} \breve{\Phi}_f(\boldsymbol{\xi}, t) + \check{c}|\boldsymbol{\xi}| \int_0^t \frac{d\tau}{\sqrt{t-\tau}} e^{-\nu\xi^2(t-\tau)} \breve{\Phi}_f(\boldsymbol{\xi}, \tau) \\ = \frac{\breve{\phi}_u(\boldsymbol{\xi}, t)}{\nu}, \quad \check{c} = -\sqrt{\nu/\pi}, \end{aligned} \tag{35}$$

for the Fourier transform $\breve{\Phi}_f$. To this purpose it is instrumental to consider the Laplace transform,

$$\mathscr{L}[\breve{\Phi}] \equiv \tilde{\Phi}(\boldsymbol{\xi}, p) = \int_0^\infty e^{-pt} \breve{\Phi}(\boldsymbol{\xi}, t)\, dt,$$

which, after use of the convolution theorem and recalling that [12]

$$\mathscr{L}\left[\frac{e^{-at}}{\sqrt{t}}\right] = \Gamma(1/2)(p+a)^{-1/2} \quad (\Gamma(1/2) = \sqrt{\pi}),$$

yields Eq. (35) in the form

$$\tilde{\Phi}_f(\boldsymbol{\xi}, p) + \check{c}|\boldsymbol{\xi}| \tilde{\Phi}_f(\boldsymbol{\xi}, p) \sqrt{\pi} (p + \nu|\boldsymbol{\xi}|^2)^{-1/2} = \frac{\tilde{\phi}_u(\boldsymbol{\xi}, p)}{\nu}$$

and, finally,

$$\tilde{\Phi}_f(\boldsymbol{\xi}, p) = \frac{\tilde{\phi}_u}{\nu} \left[1 + \frac{\sqrt{\nu}|\boldsymbol{\xi}|}{\sqrt{p + \nu|\boldsymbol{\xi}|^2} - \sqrt{\nu}|\boldsymbol{\xi}|}\right].$$

From this result, by using again the convolution theorem and recalling that [12]

$$\mathscr{L}^{-1}[(p^{1/2} + a)^{-1}] = \frac{1}{\sqrt{\pi t}} - a e^{a^2 t} \operatorname{erfc}(a\sqrt{t}),$$

so that

$$\begin{aligned} \mathscr{L}^{-1}[(\sqrt{p + \nu|\boldsymbol{\xi}|^2} - |\boldsymbol{\xi}|\sqrt{\nu})^{-1}] = \frac{1}{\sqrt{\pi t}} e^{-\nu\xi^2 t} \\ + \sqrt{\nu}|\boldsymbol{\xi}| \operatorname{erfc}(-|\boldsymbol{\xi}|\sqrt{\nu t}), \end{aligned}$$

we may obtain the solution of Eq. (35) as

$$\begin{aligned} \breve{\Phi}_f(\boldsymbol{\xi}, t) &= \frac{\breve{\phi}_u}{\nu}(\boldsymbol{\xi}, t) + \sqrt{\nu}|\boldsymbol{\xi}| \int_0^t d\tau\, \mathscr{G}(\boldsymbol{\xi}, t-\tau) \frac{\breve{\phi}_u}{\nu}(\boldsymbol{\xi}, \tau) \\ \mathscr{G}(\boldsymbol{\xi}, t) &= \frac{e^{-\nu\xi^2 t}}{\sqrt{\pi t}} + \sqrt{\nu}|\boldsymbol{\xi}| \operatorname{erfc}(-|\boldsymbol{\xi}|\sqrt{\nu t}). \end{aligned} \tag{36}$$

APPENDIX C

We show how, by using the assumptions (32) on the initial vorticity field $\boldsymbol{\omega}_0$, we obtain for

$$\int_{\mathbb{R}^2} |\boldsymbol{\xi}|^d |\breve{\phi}_u|\, d\boldsymbol{\xi}, \quad d = 2, 3,$$

the time behavior indicated in (30). To this purpose, the link between the field vorticity ensuing from the diffusion of the initial field $\boldsymbol{\omega}_0$ and the wall vortex sheet is given by Eq. (15). After accounting for the symmetry of the field resulting from the diffusion of $\boldsymbol{\omega}_0$ and considering that (Section 5) $\boldsymbol{n} \times \boldsymbol{u}_S = \boldsymbol{n} \times \nabla\phi_u$, this equation may be recast in the form

$$\mathscr{W}\phi_u = -\frac{1}{2} \boldsymbol{n}^* \cdot \nabla_\pi^* \times \int_{\mathbb{R}^3} \left(\int_{\mathbb{R}_3} \boldsymbol{\omega}_0 F_0\, dV\right) g\, d\boldsymbol{x} \quad (\boldsymbol{x}^* \in \pi), \tag{37}$$

where $\mathscr{W}\phi_u := \boldsymbol{n}^* \cdot \nabla^* \times \int_\pi g(\boldsymbol{n} \times \nabla_\pi)\phi_u\, dS = \Delta_\pi^* \int_\pi g\phi_u\, dS$. As anticipated in Section 7, for general initial conditions, $\boldsymbol{\omega}_0$, the vortex sheet at the wall may be arbitrarily strong. However, if the initial field is such to induce at $t = 0$ a zero velocity at the wall,

$$\nabla^* \times \int_{\mathbb{R}^3} \boldsymbol{\omega}_0 g\, d\boldsymbol{x} = 0 \quad \text{for } \boldsymbol{x}^* \in \pi, \tag{38}$$

the slip at the boundary after diffusion of the initial field is weak, in fact vanishing with t, as will be shown in the following. We may exploit this condition on $\boldsymbol{\omega}_0$ by rewriting Eq. (37) as

$$\mathscr{W}\,\phi_u = -\frac{1}{2}\,\boldsymbol{n}^Y\cdot\nabla_\pi^Y\times\int_{\mathbb{R}^3}\left[\left(\int_{\mathbb{R}_3}\boldsymbol{\omega}_0\,F_0\,dV\right)-\boldsymbol{\omega}_0\right]g\,d\boldsymbol{x}\quad(\boldsymbol{Y}\in\pi). \tag{39}$$

Here we note that only the components parallel to the wall of both $\int_{\mathbb{R}_3}\boldsymbol{\omega}_0\,F_0\,dV$ and $\boldsymbol{\omega}_0$ actually enter the equation. Moreover, these components are even extensions, Eq. (7), to the whole $\mathbb{R}^3$ of the corresponding components of the physical field, which is confined to the upper half-space.

From the Fourier transform of Eq. (39) we obtain, in fact, information concerning the time behavior of the quantities appearing in (30). In particular, the symbol of the operator $\mathscr{W}$ is readily estimated as $|\boldsymbol{\xi}|^2c/|\boldsymbol{\xi}| = c|\boldsymbol{\xi}|$, where $\boldsymbol{\xi}$ is the two-dimensional wave-vector. The factor $|\boldsymbol{\xi}|^2$ arises from Δ_π, while the factor $1/|\boldsymbol{\xi}|$ comes from the two-dimensional Fourier transform of g; see Appendix A, Eq. (34). By noting a contribution proportional to $|\boldsymbol{\xi}|$ at the right-hand side of the equation originating from the operator $\boldsymbol{n}\cdot\nabla\times$, we directly obtain the estimate

$$|\check{\phi}_u(\boldsymbol{\xi},\tau)|\le c\left|\int_{\mathbb{R}^2}d\boldsymbol{Y}e^{-\imath\boldsymbol{\xi}\cdot\boldsymbol{Y}}\int_{\mathbb{R}}dz\int_{\mathbb{R}^2}d\boldsymbol{X}g(\boldsymbol{Y}-\boldsymbol{x})\left[\int_{\mathbb{R}^3}F_3(\boldsymbol{x}-\boldsymbol{x}',\tau)\boldsymbol{\omega}_0(\boldsymbol{x}')\,d\boldsymbol{x}'-\boldsymbol{\omega}_0(\boldsymbol{x})\right]\right|,$$

where we denoted by $\boldsymbol{X}$ (and by $\boldsymbol{X}'$ in the following) the projection onto π of the point $\boldsymbol{x}$ (and $\boldsymbol{x}'$ respectively), while z (and z') denotes the corresponding distance from the plane, i.e., $\boldsymbol{x} = (\boldsymbol{X}, z)$. Since

$$F_3(\boldsymbol{x}-\boldsymbol{x}',\tau) = F_1(z-z',\tau)\,F_2(\boldsymbol{X}-\boldsymbol{X}',\tau),$$

we have

$$|\check{\phi}_u(\boldsymbol{\xi},\tau)|\le c\left|\int_{\mathbb{R}^2}dz\,dz'[g\star(F_2\star\boldsymbol{\omega}_0)]^{\check{}}\,F_1(z-z',\tau) - \int_{\mathbb{R}}dz[(g\star\boldsymbol{\omega}_0)]^{\check{}}\right|,$$

where $[q]^{\check{}}$ is the two-dimensional Fourier transform otherwise indicated as $\check{q}$ and $a\star b$ is the two-dimensional convolution in the plane π. By use of the convolution theorem, the right-hand side of the equation may be expressed in terms of the two-dimensional Fourier transforms $\check{g}$, Eq. (33), $\check{\boldsymbol{\omega}}_0$ and

$$\check{F}_2 = 1/(2\pi)e^{-\nu|\boldsymbol{\xi}|^2\tau} \tag{40}$$

and we obtain

$$|\check{\phi}_u(\boldsymbol{\xi},\tau)|\le c\left|\int_{\mathbb{R}}dz\;\check{g}(\boldsymbol{\xi},z)\left[\frac{1}{2\pi}\check{\boldsymbol{\omega}}_0(\boldsymbol{\xi},\tau) - \check{F}_2(\boldsymbol{\xi},\tau)\int_{\mathbb{R}}dz'\,F_1(z-z',\tau)\check{\boldsymbol{\omega}}_0(\boldsymbol{\xi},z')\right]\right|, \tag{41}$$

where the factor $1/(2\pi)$ in the first term inside the brackets follows from the convolution theorem. Since, as already mentioned, only the components of the vorticity on the plane π, $\boldsymbol{\omega}_{0_\pi}$, are effective, Eq. (41) may be written as

$$|\check{\phi}_u(\boldsymbol{\xi},\tau)|\le c\left|\int_{-\infty}^{\infty}dz\;\check{\boldsymbol{\omega}}_{0_\pi}(\boldsymbol{\xi},z)\frac{e^{-|\boldsymbol{\xi}|\,|z|}}{|\boldsymbol{\xi}|} - \frac{1}{\sqrt{4\pi\nu\tau}}\int_{-\infty}^{\infty}dz'\,\check{\boldsymbol{\omega}}_{0_\pi}(\boldsymbol{\xi},z')e^{-\nu|\boldsymbol{\xi}|^2\tau}\cdot\frac{2}{|\boldsymbol{\xi}|}\int_0^\infty dz e^{-(z-z')^2/4\nu\tau}e^{-|\boldsymbol{\xi}|\,|z|}\right|. \tag{42}$$

Here the inner integral in the variable z, originally spanning over $\mathbb{R}$, has been expressed as twice the corresponding one with z ranging from zero to infinity, owing to the symmetry property of both the field $\boldsymbol{\omega}_{0_\pi}(\boldsymbol{\xi},z')$ and the kernels involved. We used the expressions (40) and (33) for $\check{F}_2$ and $\check{g}$, respectively, and also introduced the explicit form for F_1,

$$F_1(z-z',\tau) = \frac{e^{-(z-z')/(4\nu\tau)}}{\sqrt{4\pi\nu\tau}}.$$

Finally, the constant $1/(8\pi^2)$ has been absorbed into c. From the known integral [12]

$$\int_0^\infty e^{-zp}e^{-z^2/4a}\,dz = \sqrt{\pi}\sqrt{a}\,\mathrm{erfc}(\sqrt{a}p)e^{ap^2}$$

we may evaluate the inner integral over z as

$$\sqrt{\pi}\sqrt{\nu\tau}e^{-z'^2/(4\nu\tau)}e^{\nu\tau(|\boldsymbol{\xi}|-z'/(2\nu\tau))^2}\mathrm{erfc}\left(\sqrt{\nu\tau}\left(|\boldsymbol{\xi}|-\frac{z'}{2\nu\tau}\right)\right).$$

Consequently, after renaming z' into z, the right-hand side of (42) results as

$$c\left|\int_{-\infty}^{\infty}\check{\boldsymbol{\omega}}_{0_\pi}(\boldsymbol{\xi},z)\frac{e^{-|\boldsymbol{\xi}|\,|z|}}{|\boldsymbol{\xi}|}\left[1-e^{-|\boldsymbol{\xi}|(z-|z|)}\mathrm{erfc}\left(\sqrt{\nu\tau}\left(|\boldsymbol{\xi}|-\frac{z}{2\nu\tau}\right)\right)\right]\right|.$$

If we now split the integral in two contribution, for $z \geq 0$ and for $z \leq 0$, respectively, and in the second one we change the variable from z into $-z$, summing up the two contributions finally yields

$$|\check{\phi}_u(\boldsymbol{\xi}, \tau)| \leq \int_0^\infty dz\, |\check{\boldsymbol{\omega}}_{0_\pi}(\boldsymbol{\xi}, z)| \frac{e^{-|\boldsymbol{\xi}|z}}{|\boldsymbol{\xi}|} B$$

$$B = \left[2 - \operatorname{erfc}\left(\sqrt{\nu\tau}\left(|\boldsymbol{\xi}| - \frac{z}{2\nu\tau}\right)\right) - e^{2|\boldsymbol{\xi}|z}\operatorname{erfc}\left(\sqrt{\nu\tau}\left(|\boldsymbol{\xi}| + \frac{z}{2\nu\tau}\right)\right)\right]. \tag{43}$$

Since for $\tau \to 0$ the two terms involving the complementary error function approach 2 and 0, respectively, B vanishes with τ. More precisely we have the asymptotic behavior for small τ,

$$\frac{e^{z^2/(4\nu\tau)B}}{\sqrt{\tau}} \simeq c\sqrt{\nu}|\boldsymbol{\xi}|e^{|\boldsymbol{\xi}|z},$$

which allows us to write

$$B \simeq c\nu\tau|\boldsymbol{\xi}|e^{|\boldsymbol{\xi}|z}\frac{e^{-z^2/(4\nu\tau)}}{\sqrt{4\pi\nu\tau}}.$$

Substituting into (43), after recalling the expression for F_1, yields

$$|\check{\phi}_u(\boldsymbol{\xi}, \tau)| \leq c\nu\tau \left| \int_{\mathbb{R}} dz F_1(z, \tau)\check{\boldsymbol{\omega}}_{0\pi}(\boldsymbol{\xi}, z)\right|,$$

where again we used symmetry to extend the integral to $\mathbb{R}$. From the result obtained we readily have

$$\begin{aligned}\int_{\mathbb{R}} d\boldsymbol{\xi}|\boldsymbol{\xi}|^d|\check{\phi}_u(\boldsymbol{\xi}, \tau)| &\leq c\nu\tau \int_{\mathbb{R}} dz F_1(z, \tau) \int_{\mathbb{R}^2} d\boldsymbol{\xi}|\boldsymbol{\xi}|^d|\check{\boldsymbol{\omega}}_{0_\pi}(\boldsymbol{\xi}, z)| \\ &\leq c\nu\tau \sup_z \left(\int_{\mathbb{R}^2} |\boldsymbol{\xi}|^d|\check{\boldsymbol{\omega}}_{0_\pi}(\boldsymbol{\xi}, z)|\, d\boldsymbol{\xi}\right) \int_{\mathbb{R}} F_1(z, \tau)\, dz \\ &= c\nu\tau \sup_z \left(\int_{\mathbb{R}^2} |\boldsymbol{\xi}|^d|\check{\boldsymbol{\omega}}_{0_\pi}(\boldsymbol{\xi}, z)|\, d\boldsymbol{\xi}\right),\end{aligned}$$

which, under assumption (32) on $\check{\boldsymbol{\omega}}_0$, for $d = 2, 3$, finally yields (30).

A procedure similar to the one illustrated above finally allows us to obtain the result for the case of an initial field which violates the no-slip condition at $t = 0$. Even in this case Eq. (37) provides the required relation between ϕ_u and $\boldsymbol{\omega}_0$. However, since under the present conditions Eq. (38) does not hold, to show that (28) follows from assumptions of the form (32) on the initial field $\boldsymbol{\omega}_0$ we may directly take the Fourier transform of Eq. (37). We do not report the explicit computations, which proceed along lines similar to those illustrated in detail for the previous case and we simply give the result that (28) follows from (32), now with $d = 0, 1, 2$.

APPENDIX D

The vorticity, as the curl of velocity, is intrinsically a solenoidal vector field. In the vorticity formulation, this property is certainly preserved when considering exact solutions. Instead, numerical approaches often provide vorticity fields which are solenoidal only within a truncation error. In the present case, we may show that the divergence of the approximate vorticity is exactly zero. To this purpose, let us consider first the exact equations. The divergence of the field $\boldsymbol{\omega}$ follows from representation (10) as

$$\nabla_* \cdot \boldsymbol{\omega}^*(\boldsymbol{x}^*, t) = -2\nu \int_0^t d\tau \int_\pi \boldsymbol{f}_\pi \cdot \nabla_* F\, dS + \int_{\mathbb{R}_3} \nabla \cdot \boldsymbol{\omega}_0 F_0\, dV.$$

After recalling that $\nabla_* F = -\nabla F$ and that $\boldsymbol{f}_{\pi 3} = 0$, integration by parts of the surface integral yields

$$\nabla_* \cdot \boldsymbol{\omega}^*(\boldsymbol{x}^*, t) = -2\nu \int_0^t d\tau \int_\pi \nabla_\pi \cdot \boldsymbol{f}_\pi F\, dS + \int_{\mathbb{R}_3} \nabla \cdot \boldsymbol{\omega}_0 F_0\, dV, \tag{44}$$

where the density $\boldsymbol{f}_\pi$ has been assumed to rapidly vanish at infinity. Hence, for an initially solenoidal field $\boldsymbol{\omega}_0$, Eq. (44) assures that $\nabla \cdot \boldsymbol{\omega} = 0$ when the surface divergence of the source $\boldsymbol{f}_\pi$ is zero. Actually, as discussed in Section 5, this requirement corresponds to $\psi_f = 0$ in the Hodge decomposition (19), since we have

$$\nabla_\pi \cdot \boldsymbol{f}_\pi = \Delta_\pi \psi_f.$$

When the slip velocity $\boldsymbol{n} \times \boldsymbol{u}_S$ is used instead of the exact source, for the divergence of the approximate vorticity field we have

$$\nabla_* \cdot \boldsymbol{\omega}^*(\boldsymbol{x}^*, t) = -2\nu \int_0^t d\tau \int_\pi \frac{\partial}{\partial \tau} \nabla_\pi \cdot (\boldsymbol{n} \times \boldsymbol{u}_S)\, F\, dS + \int_{\mathbb{R}_3} \nabla \cdot \boldsymbol{\omega}_0 F_0\, dV. \tag{45}$$

Since (Section 5) $\psi_u = 0$, from decomposition (18) for $\boldsymbol{u}_S$ we have

$$\nabla_\pi \cdot (\boldsymbol{n} \times \boldsymbol{u}_S) = \nabla_\pi \cdot (\boldsymbol{n} \times \nabla_\pi \phi_u) = 0,$$

and the approximate vorticity follows as an exact solenoidal field.

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